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European Journal
of Combinatorics

European Journal of Combinatorics 24 (2003) 33–43

www.elsevier.com/locate/ejc

A study of Eulerian numbers by means of an operator on permutations

Shinji Tanimoto

Department of Mathematics, Kochi Joshi University, Kochi 780-8515, Japan

Received 5 November 2001; accepted 27 September 2002

Abstract

In a previous paper an operator on permutations was defined and its application was discussed. The operator preserves the numbers of their ascents, and each permutation has its own period and orbit under the operator, by which it enables us to study Eulerian numbers. The objective of this paper is to investigate the numbers of orbits and permutations in more detail and to discuss their applications to Eulerian numbers. They include representations of Eulerian numbers by means of orbits and some congruence relations modulo prime powers. © 2003 Elsevier Science Ltd. All rights reserved.

1. Introduction

In [7] an operator σ on the set of permutations of $\{1, 2, \dots, n\}$, or the symmetric group of degree n , was defined and fundamental properties concerning the periodicity of permutations under the operator were investigated. Its applications were given to proofs of congruence relations modulo a prime for Eulerian numbers. The proofs were simple and straightforward, and needed no identities of Eulerian numbers nor theorems in number theory.

The operator σ is defined by adding one to all entries of a permutation $A = a_1 a_2 \cdots a_n$ and by changing $n + 1$ into one. However, when n appears at either end of a permutation, it is removed and one is put at the other end. That is, $\sigma(n a_2 \cdots a_n) = b_2 b_3 \cdots b_n 1$, where $b_i = a_i + 1$ for $2 \leq i \leq n$, and $\sigma(a_1 \cdots a_{n-1} n) = 1 b_1 b_2 \cdots b_{n-1}$, where $b_i = a_i + 1$ for $1 \leq i \leq n - 1$. If n does not appear at either end, $\sigma(a_1 a_2 \cdots a_n) = b_1 b_2 \cdots b_n$, where $b_i = a_i + 1$ for $1 \leq i \leq n$ and $n + 1$ is replaced by one at the position. We define successive applications of σ by $\sigma^l A = \sigma(\sigma^{l-1} A)$ for positive integers l , σ^0 being the identity operator on permutations.

An *ascent* (*descent*) in $A = a_1 a_2 \cdots a_n$ is an adjacent pair such that $a_i < a_{i+1}$ ($a_i > a_{i+1}$) for some i ($1 \leq i \leq n - 1$). For $0 \leq k \leq n - 1$, let us denote by $E(n, k)$ the

E-mail address: tanimoto@cc.kochi-wu.ac.jp (S. Tanimoto).

set of all permutations of $\{1, 2, \dots, n\}$ with exactly k ascents and by $e(n, k)$ its cardinality, the *Eulerian number*:

$$e(n, k) = |E(n, k)|.$$

For convenience sake we divide $E(n, k)$ into two classes. By $E^-(n, k)$ we denote the set of all permutations $A = a_1 a_2 \cdots a_n \in E(n, k)$ with $a_1 < a_n$ and by $E^+(n, k)$ the set of those with $a_1 > a_n$. It is easy to see that $A \in E^-(n, k)$ if and only if $\sigma A \in E^-(n, k)$ and $A \in E^+(n, k)$ if and only if $\sigma A \in E^+(n, k)$. It was shown in [7] that for a permutation A there exists a smallest positive integer $\pi(A)$ such that $\sigma^{\pi(A)} A = A$, which is called the *period* of A . Its trace

$$\{\sigma A, \sigma^2 A, \dots, \sigma^{\pi(A)} A = A\}$$

is called the *orbit* of A . Also there it was shown that the period satisfies the relation:

$$\pi(A) = \begin{cases} (n - k) \gcd(n, \pi(A)) & \text{if } A \in E^-(n, k), \\ (k + 1) \gcd(n, \pi(A)) & \text{if } A \in E^+(n, k). \end{cases} \quad (1)$$

For $A = a_1 a_2 \cdots a_n$, we define its reflection by

$$A^* = a_n \cdots a_2 a_1.$$

The reflection enjoys the easily verified properties:

- (a) $A \in E(n, k)$ if and only if $A^* \in E(n, n - k - 1)$ or, more precisely, $A \in E^-(n, k)$ if and only if $A^* \in E^+(n, n - k - 1)$;
- (b) $\sigma(A^*) = (\sigma A)^*$ and $\pi(A^*) = \pi(A)$ for all permutations A .

Accordingly, we can mainly restrict our attention to permutations in $E^-(n, k)$.

It follows from (1) that the period of a permutation $A \in E(n, k)$ is either $d(n - k)$ or $d(k + 1)$ for a positive divisor d of n , i.e.

$$d = \gcd(n, \pi(A)),$$

although there may be no permutations having such periods for some divisors. In this paper divisors of n always mean positive divisors.

As is easily shown, the orbit of a permutation in $E^-(n, k)$ contains at least one permutation of the form $A = 1a_1 \cdots a_{n-1}$ and that of a permutation in $E^+(n, k)$ contains at least one permutation of the form $A = a_1 \cdots a_{n-1}1$, where $a_1 a_2 \cdots a_{n-1}$ is a permutation of $\{2, 3, \dots, n\}$. In order to count orbits in $E^-(n, k)$ and $E^+(n, k)$, it is both sufficient and convenient to deal with these permutations. So we will call permutations of the forms

$$A = \begin{cases} 1a_1 \cdots a_{n-1} & \text{in } E^-(n, k), \\ a_1 \cdots a_{n-1}1 & \text{in } E^+(n, k) \end{cases}$$

canonical ones.

It is obvious that the cardinalities of all canonical permutations in $E^-(n, k)$ and $E^+(n, k)$ are $e(n - 1, k - 1)$ and $e(n - 1, k)$, respectively. Remarking that orbits of the same period either coincide or are disjoint and that orbits of different periods are disjoint, we can classify all canonical permutations by means of orbits.

In Section 2 we derive two kinds of representations of Eulerian numbers from the classifications of $E^-(n, k)$ and $E^+(n, k)$ by means of orbits, from which a well-known recurrence relation of Eulerian numbers follows. In Section 3 we will investigate the relationships between the numbers of orbits. In Section 4 we give a condition for periods that guarantees nonexistence of permutations and orbits with those periods satisfying it. Using the condition, we will provide a simple rule to calculate the periods of permutations. Tables exhibiting the numbers of orbits are also given for small n . In Section 5 we proceed to apply these results to congruence relations modulo prime powers. The proofs need neither analytic calculations involving identities nor properties for Eulerian numbers that are found in [1–6], for instance.

2. Representations of Eulerian numbers by means of orbits

Referring to (1), for each divisor d of n , we denote by α_d^k the number of orbits of period $d(n-k)$ in $E^-(n, k)$ and by β_d^k that of orbits of period $d(k+1)$ in $E^+(n, k)$. In particular, from [7, Theorem 7] we know the values for $d = 1$:

$$\alpha_1^k = \begin{cases} 1 & \text{if } \gcd(n, k) = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \beta_1^k = \begin{cases} 1 & \text{if } \gcd(n, k+1) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Our first theorem manifests a significance of orbits in studying Eulerian numbers and it plays a fundamental role in the subsequent investigations.

Theorem 1. *Let n and k be positive integers satisfying $1 \leq k \leq n-1$. The following representations of Eulerian numbers by means of orbits are possible:*

$$e(n-1, k-1) = \sum_{d|n} d\alpha_d^k; \quad (2)$$

$$e(n-1, k) = \sum_{d|n} d\beta_d^k; \quad (3)$$

$$e(n, k) = \sum_{d|n} d\{(n-k)\alpha_d^k + (k+1)\beta_d^k\}. \quad (4)$$

Proof. First we deal with the set of all permutations in $E^-(n, k)$, among which there are $e(n-1, k-1)$ canonical permutations. It follows from (1) that the period of a permutation $A \in E^-(n, k)$ is equal to $d(n-k)$ for a divisor d of n . So suppose $\pi(A) = d(n-k)$ for $A \in E^-(n, k)$. By [7, Corollary 2] there exist n canonical permutations in $\{\sigma A, \sigma^2 A, \dots, \sigma^{n(n-k)} A = A\}$. Hence each orbit $\{\sigma A, \sigma^2 A, \dots, \sigma^{d(n-k)} A = A\}$ of A with period $d(n-k)$ contains exactly d canonical permutations, because the latter repeats itself n/d times in the former. Since there exist α_d^k orbits of period $d(n-k)$ for each divisor d of n , classifying all canonical permutations of $E^-(n, k)$ into orbits leads us to (2).

The proof of (3) is similar. To do this we consider all permutations in $E^+(n, k)$. From (1) the period of a permutation $A \in E^+(n, k)$ is equal to $d(k+1)$ for a divisor d of n . Suppose $\pi(A) = d(k+1)$ for $A \in E^+(n, k)$. By [7, Corollary 2] there exist n canonical permutations in $\{\sigma A, \sigma^2 A, \dots, \sigma^{n(k+1)} A = A\}$ and hence, as earlier, there exist exactly d

such permutations in each orbit $\{\sigma A, \sigma^2 A, \dots, \sigma^{d(k+1)} A = A\}$ of A with period $d(k+1)$. The set of all canonical permutations in $E^+(n, k)$ has cardinality $e(n-1, k)$. Since there exist β_d^k orbits of period $d(k+1)$ for each divisor d of n , its classification by means of orbits yields (3).

Next we proceed to enumerate permutations. By counting permutations via the numbers of orbits and periods, we see that the cardinalities of $E^\pm(n, k)$ are given by

$$|E^-(n, k)| = \sum_{d|n} d(n-k)\alpha_d^k \quad \text{and} \quad |E^+(n, k)| = \sum_{d|n} d(k+1)\beta_d^k. \quad (5)$$

Since the set $E(n, k)$ is a disjoint union of $E^-(n, k)$ and $E^+(n, k)$, we conclude that

$$e(n, k) = \sum_{d|n} d(n-k)\alpha_d^k + \sum_{d|n} d(k+1)\beta_d^k,$$

which proves (4). \square

Seemingly we could expect to obtain the numbers of orbits, α_d^k or β_d^k , by applying the Möbius inversion formula to (2) or (3). Unfortunately, however, since these numbers depend on n , we realize that the Möbius inversion formula cannot be applied.

If p is a prime and m is a positive integer, then by equality (4) we can obtain, for all k ($0 \leq k \leq p^m - 1$),

$$e(p^m, k) \equiv 1 \pmod{p}.$$

It can be proved by examining the values of α_1^k and β_1^k , as was done in [7, Theorem 10]. This congruence relation is a generalization of [6, Theorem 4.1].

Making use of (2) and (3), we see that both cardinalities in (5) can be written simply by Eulerian numbers.

Corollary 2. *The cardinalities of $E^\pm(n, k)$ are given as follows:*

- (i) $|E^-(n, k)| = (n-k)e(n-1, k-1)$ for $1 \leq k \leq n-1$;
- (ii) $|E^+(n, k)| = (k+1)e(n-1, k)$ for $0 \leq k \leq n-2$.

From these equalities we can derive the well-known recurrence relation for Eulerian numbers:

$$e(n, k) = (n-k)e(n-1, k-1) + (k+1)e(n-1, k). \quad (6)$$

Our approach is quite different from [2], although not so elementary.

3. Proofs of the equalities $\alpha_d^k = \beta_d^{k-1}$ and $\alpha_d^k = \alpha_d^{n-k}$

We investigate relations for the numbers of orbits introduced in the preceding section. First we deal with relations between the numbers of both kinds of orbits. In view of (2) and (3), we may infer that the equality

$$\alpha_d^k = \beta_d^{k-1} \quad (7)$$

holds for all k and divisors d of n . The main purpose of this section is to prove that it is indeed true. To do this, for $A = a_1 a_2 \cdots a_n$, we define its left-shift by

$$A^{\natural} = a_2 a_3 \cdots a_n a_1.$$

In order to investigate the period and orbit of the left-shift, it is useful to consider the inverse operator σ^{-1} of σ . It operates on a permutation $A = a_1 a_2 \cdots a_n$ as follows. Each entry of a permutation is subtracted by one and one is changed into n . However, when one appears at either end of the permutation, it is deleted and n is placed at the opposite end. For example, $\sigma^{-1}(351\ 624) = 246\ 513$, $\sigma^{-1}(153\ 246) = 421\ 356$ and $\sigma^{-1}(534\ 621) = 642\ 351$. It is easy to see that permutations have the same periods and orbits under σ^{-1} as those under σ and that $A \in E^-(n, k)$ if and only if $\sigma^{-1}A \in E^-(n, k)$, and $A \in E^+(n, k)$ if and only if $\sigma^{-1}A \in E^+(n, k)$. When we call orbits under σ forward orbits, those under σ^{-1} become backward orbits.

Theorem 3. *Let n and k be positive integers satisfying $1 \leq k \leq n-1$ and let d be a divisor of n . Then the number of orbits of period $d(n-k)$ in $E^-(n, k)$ equals that of orbits of period dk in $E^+(n, k-1)$, that is, $\alpha_d^k = \beta_d^{k-1}$.*

Proof. Suppose the period of $A \in E^-(n, k)$ is $d(n-k)$ for a divisor d of n . As in the proof of Theorem 1, there exist d canonical permutations in the orbit of A : $\{\sigma A, \sigma^2 A, \dots, \sigma^{d(n-k)} A = A\}$. Now let $B = 1b_2b_3 \cdots b_n$ be any canonical permutation in the orbit. Then it follows that

$$\sigma^{n+1-b_n} B = 1c_1c_2 \cdots c_{n-1}$$

by definition of σ , where $c_1 = 1 + (n+1-b_n)$ and, for $2 \leq i \leq n-1$, $c_i = b_i + (n+1-b_n)$ if $b_i \leq b_n - 1$ and $c_i = b_i + 1 - b_n$ if $b_i > b_n - 1$.

On the other hand, we have $B^{\natural} = b_2b_3 \cdots b_n 1 \in E^+(n, k-1)$, and by definition of σ^{-1} we get $\sigma^{-1}B^{\natural} = n(b_2-1) \cdots (b_n-1)$. Further b_n-2 times applications of σ^{-1} yield

$$\sigma^{-(b_n-1)} B^{\natural} = c'_1 c'_2 \cdots c'_{n-1} 1,$$

where $c'_1 = n - (b_n - 2)$ and, for $2 \leq i \leq n-1$, $c'_i = n + b_i - 1 - (b_n - 2)$ if $b_i \leq b_n - 1$ and $c'_i = b_i - 1 - (b_n - 2)$ if $b_i > b_n - 1$.

Consequently, $c_1c_2 \cdots c_{n-1}$ and $c'_1c'_2 \cdots c'_{n-1}$ are the same permutation of $\{2, 3, \dots, n\}$. Thus $1c_1c_2 \cdots c_{n-1}$ belongs to the orbit of B (or A) and $c_1c_2 \cdots c_{n-1}1$ belongs to that of B^{\natural} . This argument also implies that the permutation $1c_1c_2 \cdots c_{n-1}$ is the first canonical one following B in the (forward) orbit and that the permutation $c_1c_2 \cdots c_{n-1}1$ is the first canonical one followed by B^{\natural} in the (backward) orbit.

Repeating this process, we see that each canonical permutation $1c_1c_2 \cdots c_{n-1}$ in the orbit of B has its counterpart $c_1c_2 \cdots c_{n-1}1$ in the orbit of B^{\natural} . If a permutation $1c_1c_2 \cdots c_{n-1}$ returns to itself in the orbit of B , then so does $c_1c_2 \cdots c_{n-1}1$ in the orbit of B^{\natural} . Therefore, we conclude that there also exist d canonical permutations in the orbit of B^{\natural} . By using (1), if we put $\pi(B^{\natural}) = lk$ for a divisor l of n , then there exist l canonical permutations in the orbit of B^{\natural} , as was shown in the proof of Theorem 1. From the above observation we conclude that $l = d$ and that the period of B^{\natural} is equal to dk .

Thus the above argument tells us that to an orbit of period $d(n-k)$ in $E^-(n, k)$ there corresponds a unique orbit of period dk in $E^+(n, k-1)$. Therefore, the number of orbits of period $d(n-k)$ in $E^-(n, k)$ is equal to that of orbits of period dk in $E^+(n, k-1)$. \square

Let $A = a_1 a_2 \cdots a_n$ be a permutation in $E^-(n, k)$. Then we have $A^\natural \in E^-(n, k)$ or $A^\natural \in E^+(n, k-1)$, according to whether $a_2 < a_1$ or $a_2 > a_1$, respectively. As for the period of A^\natural we can obtain the next result by an analogous technique.

Corollary 4. *Let $A \in E^-(n, k)$. If A^\natural belongs to $E^-(n, k)$, then A and A^\natural have the same orbit and hence $\pi(A^\natural) = \pi(A)$. If A^\natural belongs to $E^+(n, k-1)$, its period is given by $\pi(A^\natural) = k\pi(A)/(n-k)$.*

Proof. The condition for the first case is $a_2 < a_1 < a_n$. Because $a_1 < a_n$ in $A^\natural = a_2 \cdots a_n a_1$, we see that σ^n moves only a_1 to the left-hand end in A^\natural ; $\sigma^n A^\natural = a_1 a_2 \cdots a_n = A$. Equivalently, we get $\sigma^{-n} A = A^\natural$. Therefore, it follows that the orbits of A^\natural and A coincide.

Next suppose $A^\natural \in E^+(n, k-1)$, i.e. $a_1 < a_2$ and $a_1 < a_n$. The permutation $\sigma^{n+1-a_n} A$ becomes $1b_1b_2 \cdots b_{n-1}$, where $b_i = a_i + (n+1-a_n)$ if $a_i \leq a_n - 1$ and $b_i = a_i - a_n + 1$ if $a_i > a_n - 1$. On the other hand, an application of σ^{-a_1} to A^\natural turns the entry a_1 of A^\natural into n at the left-hand end of the permutation; $\sigma^{-a_1} A^\natural = na'_2 \cdots a'_n$ and $a'_n = a_n - a_1$. We can apply σ^{-1} ($a_n - 1 - a_1$) more times to A^\natural , for we have $a_n - 1 \geq a_1$ by assumption. Then finally we get $\sigma^{-(a_n-1)} A^\natural = c_1 c_2 \cdots c_{n-1} 1$. Here, for each i ($1 \leq i \leq n-1$), $c_i = n + a_i - (a_n - 1)$ if $a_i \leq a_n - 1$ and $c_i = a_i - (a_n - 1)$ if $a_i > a_n - 1$. Therefore, we see that $b_i = c_i$ for all i ($1 \leq i \leq n-1$).

If we denote the period of the canonical permutation $\sigma^{n+1-a_n} A$ by $d(n-k)$ for some divisor d of n , then that of the canonical permutation $\sigma^{-(a_n-1)} A^\natural$ is equal to dk from the same argument of the proof of Theorem 3. Hence we obtain

$$(n-k)\pi(\sigma^{-(a_n-1)} A^\natural) = k\pi(\sigma^{n+1-a_n} A).$$

Remarking that $\pi(\sigma^{-(a_n-1)} A^\natural) = \pi(A^\natural)$ and $\pi(\sigma^{n+1-a_n} A) = \pi(A)$, the required equality follows. \square

The rest of this section is devoted to a symmetry property for α_d^k . An immediate application of reflected permutations leads us to a well-known symmetry property for Eulerian numbers

$$e(n, k) = e(n, n-k-1),$$

and to one for α_d^k described in the next corollary.

Corollary 5. *The numbers α_d^k satisfy the equality*

$$\alpha_d^k = \alpha_d^{n-k}$$

for $1 \leq k \leq n-1$ and divisors d of n . Thus α_d^k can also be looked upon as the number of orbits of period dk in $E^-(n, n-k)$.

Proof. Applying properties (a) and (b) of reflected permutations in Section 1, it can be shown that to the orbit of a permutation $A \in E^-(n, k)$ with period $d(n-k)$ there uniquely corresponds that of $A^* \in E^+(n, n-k-1)$ with the same period. Therefore, we get $\alpha_d^k = \beta_d^{n-k-1}$ and, moreover, $\beta_d^{n-k-1} = \alpha_d^{n-k}$ from (7). This completes the proof. \square

4. Orbits and periods of permutations

For $2 \leq k \leq n-2$ we see that $\alpha_n^k \geq 1$, since there exists always a permutation with maximal period $n(n-k)$ in $E^-(n, k)$, e.g. $A = 12 \cdots kn(n-1) \cdots (k+1)$. On the other hand, we can give a condition under which α_d^k is equal to zero. It turns out that the value $\gcd(k, n/d)$ plays an important role in it. It is also used to provide a simple procedure to compute the period of a given permutation.

Theorem 6. *If a divisor d of n is larger than one, then $\alpha_d^1 = \alpha_d^{n-1} = 0$. Moreover, if $2 \leq k \leq n-2$ and $\gcd(k, n/d) > 1$, then $\alpha_d^k = 0$.*

Proof. The proof for $\alpha_d^{n-1} = \alpha_d^1 = 0$ ($d > 1$) is as follows. For $k = n-1$, we see that $A = 12 \cdots n$ is a unique permutation in $E^-(n, n-1)$ and that its period is one. Therefore, we get $\alpha_1^{n-1} = 1$ and $\alpha_d^{n-1} = 0$ for divisors d ($d > 1$). Similarly, since $A = n \cdots 21$ is a unique permutation in $E^+(n, 0)$ and has a period one, we obtain $\alpha_1^1 = \beta_1^0 = 1$ and $\alpha_d^1 = \beta_d^0 = 0$ for divisors d ($d > 1$).

Next suppose $2 \leq k \leq n-2$. The period of a permutation $A \in E^-(n, k)$ must satisfy $\pi(A) = (n-k) \gcd(n, \pi(A))$ from (1). Putting $d = \gcd(n, \pi(A))$, we have $\pi(A) = d(n-k)$ and $d = \gcd(n, d(n-k))$, which implies $\gcd(n-k, n/d) = 1$ or $\gcd(k, n/d) = 1$. Consequently, we see that there exist no permutations of period $d(n-k)$, i.e. $\alpha_d^k = 0$, if $\gcd(k, n/d) > 1$. \square

As an application of this theorem we are able to provide a simple rule for calculating the periods of permutations. Let $A = a_1 a_2 \cdots a_n$ be a permutation of $E^-(n, k)$, where $1 \leq k \leq n-2$. We call a consecutive sequence $a_i \cdots a_j$ in $a_1 a_2 \cdots a_n$ an *ascending chain* if $a_{i-1} > a_i < a_{i+1} < \cdots < a_j > a_{j+1}$, where a_0 or a_{n+1} is discarded when it appears. The number of ascending chains is determined by the number of descents, which is equal to $n-k-1$. Since ascending chains are separated by these $(n-k-1)$ descents, we can write A as

$$A = C_1 C_2 \cdots C_{n-k},$$

by arranging $n-k$ ascending chains of A .

Let d be the smallest positive integer such that

$$\sigma^d A = C_{n-k-q+1} \cdots C_{n-k} C_1 \cdots C_{n-k-q}$$

for some positive integer q , where $1 \leq q \leq n-k-1$. We already know that $1 \leq d \leq n$, because

$$\sigma^n A = C_{n-k} C_1 C_2 \cdots C_{n-k-1} \quad \text{and} \quad \sigma^{-n} A = C_2 \cdots C_{n-k-1} C_{n-k} C_1 \quad (8)$$

were proved in [7, Theorem 1]. Namely, each application of σ^n always moves the last ascending chain to the head and hence that $\sigma^{n(n-k)}A = A$. Then, by using this d , the period of A is given by

$$\pi(A) = d(n-k).$$

In what follows we show that $d(n-k)$ is indeed the period of A . First we observe that, using (8) several times,

$$\sigma^d A = \sigma^{qn} A$$

holds, which implies $\sigma^{qn-d}A = A$ and hence $\sigma^{(qn-d)(n-k)}A = A$. Noting $\sigma^{qn(n-k)}A = A$, it follows that $\sigma^{d(n-k)}A = A$. Next let us suppose $\pi(A) = d'(n-k)$ for some divisor d' of n . Then we get $d' \leq d$. By Theorem 6, we have $\gcd(n-k, n/d') = 1$, which implies that there exist two integers s and t such that $s(n-k) + tn/d' = 1$ or $d' = sd'(n-k) + tn$. By the assumption $\Sigma^{d'(n-k)}A = A$ this implies $\sigma^{d'}A = \sigma^{tn}A$. Note that d' is not the period of A , because we are considering the case $1 \leq k \leq n-2$. Using (8) again, we see that $\sigma^{d'}A$ becomes a permutation of the form $C_{n-k-q+1} \cdots C_{n-k} C_1 \cdots C_{n-k-q}$ for some positive integer q ($1 \leq q \leq n-k-1$). Therefore, by the assumption on d we get $d \leq d'$ and hence we conclude that both are equal. This proves our assertion.

As for the period of $A \in E^+(n, k)$, it suffices to apply this rule to $A^* \in E^-(n, n-k-1)$, because $\pi(A) = \pi(A^*)$.

Example 1. Set $a = 10$, $b = 11$, $c = 12$. We consider a permutation $A = 24c5738a6b19 \in E^-(12, 7)$ and denote its expression in terms of ascending chains by $A = 24c.57.38a.6b.19$. The following table exhibits the expressions of σA , $\sigma^2 A$ and $\sigma^3 A$ in terms of ascending chains.

σA	$\sigma^2 A$	$\sigma^3 A$
35.168.49b.7c.2a	46.279.5ac.8.13b	57.38a.6b.19.24c

In $\sigma^3 A$ the same ascending chains as in A occur cyclically. Therefore, $\pi(A)$ is not $12 \times (12-7) = 60$, but $3 \times (12-7) = 15$. \square

The following tables give the values of α_d^k for $6 \leq n \leq 10$. These show that most permutations have maximal periods $n(n-k)$. The values α_1^k are described at the beginning of Section 2 and all zeros follow from Theorem 6.

TABLES OF α_d^k

$n = 6$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$d = 1$	1	0	0	0	1
$d = 2$	0	1	0	1	0
$d = 3$	0	0	2	0	0
$d = 6$	0	4	10	4	0

$n = 7$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$d = 1$	1	1	1	1	1	1
$d = 7$	0	8	43	43	8	0

$n = 8$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
$d = 1$	1	0	1	0	1	0	1
$d = 2$	0	0	1	0	1	0	0
$d = 4$	0	0	5	0	5	0	0
$d = 8$	0	15	146	302	146	15	0

$n = 9$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$
$d = 1$	1	1	0	1	1	0	1	1
$d = 3$	0	1	0	4	4	0	1	0
$d = 9$	0	27	477	1734	1734	477	27	0

$n = 10$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$
$d = 1$	1	0	1	0	0	0	1	0	1
$d = 2$	0	1	1	2	0	2	1	1	0
$d = 5$	0	0	15	0	46	0	15	0	0
$d = 10$	0	50	1453	8823	15596	8823	1453	50	0

5. Eulerian numbers modulo prime powers

Some congruence relations for Eulerian numbers modulo a prime were investigated by the operator in a previous paper [7]. In this section we apply (2), (3) or (4) to a further study of novel properties for Eulerian numbers. Our proofs are straightforward and need no analytic (and complicated) calculations involving identities of Eulerian numbers that can be found in [1–6], for example.

Although either (2) or (3) can be used in proving properties of Eulerian numbers, the latter may be more useful than the former, for the expression of the period becomes simpler. The next theorem provides a condition under which Eulerian numbers are divisible by prime powers. It is obvious that Theorem 6 is also closely related to the condition.

Theorem 7. *Let p be a prime. Suppose n is divisible by p^m for a positive integer m . If k is divisible by p , then $e(n-1, k-1)$ is also divisible by p^m .*

Proof. We deal with permutations A in $E^+(n, k-1)$ rather than those in $E^-(n, k)$. Then from (1) the equation for $\pi(A)$ is described by the following simpler form:

$$\pi(A) = k \gcd(n, \pi(A)). \quad (9)$$

Without loss of generality we can assume that m is the largest integer for which p^m divides n . In what follows all integers appearing in front of powers of p are assumed to be positive integers relatively prime to p . Let k be a multiple of p .

First assume that $k = h_1 p^i$ for $1 \leq i < m$. If we set $\gcd(n, \pi(A)) = h_2 p^j$ for some j ($0 \leq j \leq m$), then (9) yields $\pi(A) = k \gcd(n, \pi(A)) = h_1 h_2 p^{i+j}$. Using $\pi(A) = h_1 h_2 p^{i+j}$ and $\gcd(n, \pi(A)) = h_3 p^{\min(m, i+j)}$, let us evaluate the integer k from the expression

$$k = \pi(A) / \gcd(n, \pi(A)).$$

According to whether $i + j \leq m$ or $i + j > m$, we have $\gcd(k, p) = 1$ or $k = h_4 p^{i+j-m}$, respectively. However, the former contradicts $\gcd(k, p) = p$ and the latter implies $j = m$, since $k = h_1 p^i$ by assumption. Therefore, we see that $\gcd(n, \pi(A)) = h_2 p^m$.

Next assume that $k = h_1 p^i$ for $i (i \geq m)$. It follows again from (9) that $\pi(A)$ is a multiple of p^m and so is $\gcd(n, \pi(A))$. Thus we also have $\gcd(n, \pi(A)) = h_2 p^m$.

The integers d in

$$e(n-1, k-1) = \sum_{d|n} d \beta_d^{k-1} \quad (10)$$

are equal to some $\gcd(n, \pi(A))$. Thus, if k is a multiple of p , we have seen that any d for which β_d^{k-1} is possibly not zero must be a multiple of p^m . Therefore, it follows from (10) that $e(n-1, k-1)$ is divisible by p^m . \square

In the proof of this theorem it should be observed that expression (9) has played the role like a recurrence relation for $\pi(A)$. When n and k are divisible by a common prime p , $\pi(A)$ obviously becomes a multiple of p from (9). If n is divisible by a higher power of p , by one more application of (9) we conclude that $\pi(A)$ is also a multiple of p^2 . Repeating this process, $\pi(A)$ turns out divisible by the highest power of p that divides n , and so is $\gcd(n, \pi(A))$. This provides us with another proof that $\gcd(n, \pi(A))$ is divisible by the highest power of p that divides n .

When k is not a multiple of p in Theorem 7, $e(n-1, k-1)$ is not necessarily divisible by p . For example, the next result summarizes the case where n is a prime power p^m .

Corollary 8. *If p^m is a prime power for a positive integer m , then*

$$e(p^m - 1, k - 1) \equiv \begin{cases} 1 \pmod{p} & \text{if } \gcd(p, k) = 1, \\ 0 \pmod{p^m} & \text{if } \gcd(p, k) = p. \end{cases}$$

Proof. We only deal with the case where $\gcd(p, k) = 1$. Equality (10) becomes

$$e(p^m - 1, k - 1) = \sum_{d|p^m} d \beta_d^{k-1},$$

and we know that $\beta_1^{k-1} = 1$, because $\gcd(p^m, k) = 1$. Thus the required congruence relation follows. \square

Under the same setting of Theorem 7 we have congruence relations for Eulerian numbers of the following type.

Corollary 9. *Let p be a prime. Suppose n is divisible by p^m for a positive integer m .*

(i) *If k is divisible by p^i for an integer $i (1 \leq i \leq m)$, then*

$$e(n, k) \equiv (k+1)e(n-1, k) \pmod{p^{m+i}}.$$

(ii) *If $k+1$ is divisible by p^i for an integer $i (i \geq 1)$, then*

$$e(n, k) \equiv (n-k)e(n-1, k-1) \pmod{p^{m+i}}.$$

Proof. For the proof of (i), we use the equality

$$e(n, k) - (k + 1)e(n - 1, k) = (n - k)e(n - 1, k - 1),$$

which is obtained from (6). Since $n - k$ is divisible by p^i , Theorem 7 implies that the right-hand side of this equality is divisible by p^{m+i} . Thus we get (i). Similarly, apply Theorem 7 to the equality

$$e(n, k) - (n - k)e(n - 1, k - 1) = (k + 1)e(n - 1, k).$$

If $k + 1$ is divisible by p^i , then the right-hand side of this equality is divisible by p^{m+i} . Thus the second relation (ii) also follows. \square

Theorem 7 tells us that it is not $e(n, k)$ but $e(n - 1, k)$ that is regularly divisible by each divisor of n as k ranges from 1.

Acknowledgements

The author is very grateful to the referees for many helpful suggestions.

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